(8 pages) **Reg. No. :** ....

Code No. : 6365 **Sub. Code : ZMAM 12**

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2022.

First Semester

Mathematics — Core

ANALYSIS — I

(For those who joined in July 2021 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. Any discrete metric space is ————
   (a) first category (b) second category
   (c) third category (d) none of these

2. Any discrete metric space having more than one point is ————
   (a) connected (b) finite
   (c) null set (d) disconnected

3. If the sequence $\{a_n\}$ is bounded and sequence $\{b_n\}$ converges to zero then the sequence $\{a_n b_n\}$ ————
   (a) diverges to $+\infty$ (b) diverges to $-\infty$
   (c) converges to zero (d) none of these

4. Find lim sup a_n for the sequence $\{a_n\} = \{n!\}$ ————
   (a) 1 (b) 0
   (c) ∞ (d) none of these

5. Applying Cauchy's root test the series $\sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^n$ is ————
   (a) convergent
   (b) divergent
   (c) neither convergent nor divergent
   (d) both convergent and divergent

6. If the n^{th} term of a series is $a_n = \frac{1.2.3\cdots n}{3.5.7\cdots 2n-1}$ then $\lim\limits_{n\to\infty}\frac{a_n}{a_{n+1}} =$ ————
   (a) 1 (b) 2
   (c) $\frac{1}{2}$ (d) 0

7. Which of the following is equivalent to compactness in a metric space M?

(a) M is totally bounded

(b) M is complete

(c) Every bounded subset of M has a limit point

(d) Every infinite subset of M has a limit point

8. Which of the following subset of R is both compact and connected? ————

(a) R (b) (0, 1)

(c) [0,100] (d) Q

9. Let f be defined on $[a, b]$; if f has a local maximum at a point $x \in (a,b)$, and if $f'(x)$ exists, then $f'(x)$ ————

(a) 1 (b) 2

(c) 0 (d) ∞

10. Suppose f is differentiable in (a, b) if $f'(x) = 0$ or all $x \in (a,b)$, then f is ————

(a) monotonically increasing

(b) monotonically decreasing

(c) constant

(d) none of these

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Show that the closed subsets of compact sets are compact.

Or

(b) Let K be a positive integer. If $\{I_n\}$ is a sequence of k-cells such that $I_n \supset I_{n+1} (n = 1, 2, 3, \cdots)$, then prove that $\bigcap_1^\infty I_n$ is not empty.

12. (a) Show that if $p > 1$, $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges; if $p \le 1$, the series diverges.

Or

(b) Suppose $\{S_n\}$ is monotonic. Then prove that $\{S_n\}$ converges iff it is bounded.

13. (a) If $\sum a_n = A$, and $\sum b_n = B$, then prove that $\sum (a_n + b_n) = A + B$, and $\sum ca_n = CA$ for any fixed c.

Or

(b) Prove :

(i) the partial sums A_n of $\sum a_n$ from a bounded sequence;

(ii) $b_0 \geq b_1 \geq b_2 \geq \cdots$;

(iii) $\lim_{n \to \infty} b_n = 0$.

14. (a) Let f be monotonic on (a, b). Then prove that the set of points of (a, b) at which f is discontinuous is at most countable.

Or

(b) Suppose f is a continuous mapping of a compact metric space X into a metric space Y then. Prove that $f(X)$ is compact.

15. (a) If f and g are continuous real function on $[a, b]$ which are differentiable in (a, b), then prove that there is a point $x \in (a, b)$ at which $[f(b) - f(a)] g'(x) = [g(b) - g(a)] f'(x)$. Note that differentiability is note required at the end points.

Or

(b) Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$. Then prove that there is a point $x \in (a, b)$ such that $f'(x) = \lambda$. A similar result holds of course if $f'(a) > f'(b)$.

PART C — (5 × 8 = 40 marks)

Answer ALL questions choosing either (a) or (b).

16. (a) Prove that a subset E of the real line R^1 is connected iff it has the following property: If $x \in E, Y \in E$, and $x < z < y$, then $z \in E$.

Or

(b) Suppose $K \subset Y \subset X$. Then prove K is compact relative to X iff K is compact relative to Y.

17. (a) Prove that the following :

(i) If $\{p_n\}$ is a sequence in a compact metrix space X, then some sub-sequence of $\{p_n\}$ converges to a point of X.

(ii) Every bounded sequence in R^k contains a convergent subsequence.

Or

(b) Prove that e is irrational

18. (a) Suppose

(i) $\sum_{n=0}^{\infty} a_n$ converges absolutely

(ii) $\sum_{n=0}^{\infty} a_n = A$,

(iii) $\sum_{n=0}^{\infty} b_n = B$,

(iv) $C_n \sum_{k=0}^{n} a_k \, b_{n-k} \; (n = 0, 1, 2, \cdots)$.

Then prove that $\sum_{n=0}^{\infty} C_n = AB$. That is, the product of two convergent series converges, and to the right value, if at least one of the two series converges absolutely.

Or

(b) State and prove Ratio Test.

19. (a) Let f be a continuous mapping of a compact metric space X into a metric space Y. Then prove that f uniformly continuous on X.

Or

(b) Let X, Y, E, f, and p is a limit point of E. Then prove that $\lim_{x \to p} f(x) = q$ iff $\lim_{n \to \infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E such that $p_n \neq p$, $\lim_{n \to \infty} p_n = p$.

20. (a) Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If $h(t) = g(f(t)) \; (a \leq t \leq b)$, then prove that h is differentiable at x, and $h'(x) = g'(f(x)) f'(x)$.

Or

(b) State and prove that L' Hospital's rule.
